EXTENSION OF THE METHODS OF LIMIT ANALYSIS TO THE INVESTIGATION OF THE PLASTIC FLOW OF BODIES WITH VARIABLE GEOMETRY

V. N. Zalesov

UDC 539.374

Making use of the fact that the limit state does not depend on the loading history, the author proposes that limit analysis theory be used to investigate plastic flow processes associated with a change in initial geometry by representing the deformation process as a continuous sequence of limit states defined by a given instantaneous configuration.

The extremum principles of rigid-plastic flow theory were developed in order to investigate the equilibrium limits of nondeformable bodies, i.e., bodies whose deformation in the limit state is so small that the equilibrium conditions can be written for the initial geometry. The idea of extending the extremum principles to the analysis of processes of plastic flow associated with distortion is based on the important observation that, as distinct from the general case of plastic deformation, the limit state does not depend on the loading history and the initial stresses. By representing the process of plastic deformation as a continuous sequence of limit states defined by a given instantaneous configuration of the deformed body, one can use this property of the limit state, on the one hand, to obtain the curve of the required load ensuring continuity of the deformation process and, on the other, to estimate the reserves of carrying capacity determined by the distortion of the initial geometry.

1. The mathematical formulation of the extremum principles of rigid-plastic flow theory is given by the system of inequalities [1]

$$\int_V Q dV \leqslant \lambda \int_S \mathbf{Pu}\, dS \leqslant \tau_0 \int_V H dV, \tag{1}$$

where

$$Q = \sigma_x \xi_x + \sigma_y \xi_y + \sigma_z \xi_z + \tau_{xy} \xi_{xy} + \tau_{yz} \xi_{yz} + \tau_{zx} \xi_{zx}$$

is the sum of the products of the statically admissible stresses $\sigma_x, \ldots, \tau_{zx}$ and the effective strain rates $\xi_x, \ldots, \xi_{zx}$;

$$H = \left\{ \frac{2}{3} \left[(\xi_x' - \xi_y')^2 + \ldots + \frac{3}{2} \left(\xi_{xy}'^2 + \xi_{yz}'^2 + \xi_{zx}'^2 \right) \right] \right\}^{1/2}$$

is the intensity of the kinematically possible strain-rate field; λ is the limit load coefficient.

In order to solve the problem of the equilibrium limit of an undeformed body it is sufficient to construct the stress field, satisfying only the equilibrium equations and the conditions on the surface S, and the strain-rate field compatible with the constraints imposed on the body. By solving inequalities (1) for λ, we obtain its upper and lower bounds

$$\lambda_u \leqslant \frac{\tau_0 \int_V H dV}{\int_S \mathbf{Pu} dS}, \quad \lambda_l \geqslant \frac{\int_V Q dV}{\int_S \mathbf{Pu} dS}.$$

Mogilev Machine-Building Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 18, No. 5, pp. 905-909, May, 1970. Original article submitted February 10, 1969.

Obviously, in this case λ_u and λ_l will be expressed as functions of the mechanical τ_0 and certain geometric q_i parameters determining the initial configuration of the body.

Proceeding to the analysis of the plastic deformation processes, we assume that the deformation kinematics are known. The construction of the plastic mechanisms and the corresponding strain-rate schemes is a problem of the limit analysis of nondeformable bodies and an object of special attention. A number of examples of the construction of the plastic mechanisms for specific structures under given loading can be found in [2].

It should be kept in mind that if the deformation mechanism has n degrees of freedom, then the analysis of the process reduces to the analysis of n independent plastic mechanisms, the instantaneous geometry of each of which is determined by a single generalized coordinate δ_n. Accordingly, without loss of generality, we can confine our attention to a deformation mechanism having only one degree of freedom determined by the coordinate δ. Together with the geometric relations the kinematics of the plastic mechanism enable us to express the variable values of the parameters q_i, describing the instantaneous configuration of the deformed body during the process of deformation, as functions of δ

$$q_i = \varphi_i(\delta).$$

Introducing φ_i into (1) and solving for λ, we obtain not the limit points, but the limit curves

$$\lambda_u^*(\delta) \leqslant \frac{\tau_0 \int\limits_V H^* dV}{\int\limits_S \mathbf{Pu} dS}, \quad \lambda_l^*(\delta) \geqslant \frac{\int\limits_V Q^* dV}{\int\limits_S \mathbf{Pu} dS} \tag{2}$$

of the load required to ensure a continuous deformation process, whose instantaneous geometry, corresponding to the assumed plastic mechanism, is determined by the variable value of δ. It is also desirable to represent the deformation load in the form of an averaged curve with an estimate of the possible range of values

$$\lambda^* = \frac{1}{2}(\lambda_u^* + \lambda_l^*) \pm \frac{1}{2}(\lambda_u^* + \lambda_l^*).$$

Obviously, the beginning of deformation corresponds to the limit state of the geometrically invariant body, when $\delta = \delta_0$ and $\lambda_{uf}^* = \lambda_B$, $\lambda_{ui}^* = \lambda_H$.

2. By taking a similar approach it is possible to refine the limit load and the actual concept of the carrying capacity of plastic structures in the region of large deformations. Whereas the carrying capacity of a nondeformable rigid-plastic structure is uniquely determined by its limit state, that of a structure operating under conditions of considerable distortion of the initial geometry remains more or less indeterminate in the areas of both analytic and experimental investigation.

In fact, consider a rigid-plastic body loaded by statically increasing loads. In the initial stage of loading, within the limits of the assumed rigid-plastic scheme, the body remains rigid. Then, starting from a certain instant, an increase in load will be accompanied by the initiation and development of plastic zones still without distortion of the initial geometry. When a certain state, usually called the limit state, is reached, the plastic zones develop to such an extent that the body is transformed into a plastic mechanism and an intense increase in plastic deformations, accompanied by considerable distortion of the initial geometry, sets in. It is natural to inquire whether the limit state always characterizes the carrying capacity of a rigid-plastic body. To answer this question it is necessary to consider the possible deformation paths. Here, there are two possible paths of plastic deformation development: the path leading to the region of increasing strains without further increase in load, in which case the limit state in fact exhausts the carrying capacity of the structure, and the path along which the development of the plastic strains under the limit loads ceases when a certain degree of distortion of the initial geometry is reached, so that further deformation is possible only if there is a corresponding increase in the acting loads beyond the limit value. The structure still functions in the post-limit state. This increase in the carrying capacity of plastic structures as a result of the distortion of the initial geometry may be described as "geometric hardening." If mechanical hardening is usually treated as an extra reserve of strength, under certain conditions geometric hardening might also be used for the same purpose. Naturally, the carrying capacity of geometrically hardening bodies remains indefinite until the permissible degree of deformation is given. This is easily done,

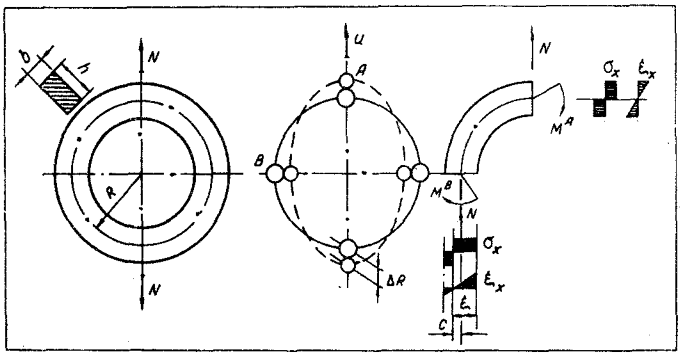

Fig. 1. Bending mechanism of a ring bent by two radial loads.

for example, by specifying the maximum permissible value of δ. By virtue of (2) the carrying capacity of deformable bodies is then given by

$$\lambda_l^*(\delta_{\max}) \leqslant \lambda \leqslant \lambda_u^*(\delta_{\max}).$$

3. As an example we will consider the problem of a circular ring of rectangular cross section bent by two radial forces. The plastic mechanism is assumed to be four-hinged (see Figure 1, where the stress and strain-rate distributions in the plastic hinges are also given). In the light of the technical theory of bending, the actual and possible strain rates and, moreover, the actual and statically admissible stresses in the rotating plastic hinges coincide and by virtue of the incompressibility condition

$$\xi_x + \xi_y + \xi_z = 0 \tag{3}$$

the bounds given by inequalities (1) approach and reduce to the single equation

$$N_u = \int_V \sigma_x \xi_x dV.$$

Introducing the rate of rotation in the hinges ω, in accordance with Figure 1 we can rewrite Eq. (3) in the form

$$2\omega(M^A + M^B) + \omega cN = \omega N(R - \Delta R + c).$$

Substituting the expressions for the moments and forces in the hinges

$$M^A = M_0 = \frac{1}{4}\sigma_0 bh^2, \quad M^B = 4M_0\xi(1-\xi),$$

$$N = N_0(2\xi - 1) = \sigma_0 bh(2\xi - 1), \quad \xi = \frac{1}{2}\left(\frac{N}{N_0} - 1\right)$$

and solving the expression obtained for N, we obtain

$$N = \frac{N_0 h}{R - \Delta R} = \frac{4M_0}{R - \Delta R}.$$

This expression gives the curve of the load N required for the continuous deformation of the ring in its own plane, determined by measuring the vertical and horizontal diameters. Specifying the maximum permissible value of this deformation ΔR, we obtain an equation for the carrying capacity of the ring under radial loading with allowance for the permissible distortion of the initial shape

$$N_{\max} = \frac{N_0 h}{R - \Delta R_{\max}}.$$

In this case, since the limit load corresponds to $\Delta R = 0$, the geometric hardening coefficient

$$k = \frac{RN_{\max}}{N_0 h} = \frac{1}{1 - \frac{\Delta R_{\max}}{R}}.$$

A more complicated example of the application of the proposed method is given in [3], where it is used to solve the plastic problem of the turning inside out of a shell of revolution.

NOTATION

x, y, z	are the coordinate axes;
V, S	are the volume and surface area;
H	is the strain rate intensity;
Q	is the power of the stress field;
$\mathbf{P}, \mathbf{u}$	are the external loads and the displacements of surface points;
σ_x	is the normal stress;
τ_{xy}	is the shear stress;
ξ_x	is the normal strain rate;
ξ_{xy}	is the shear strain rate;
τ_0	is the shear yield stress;
λ_u, λ_l	are the upper and lower values of the limit load coefficient;
q_i, δ	are the geometric parameters and generalized coordinate;
N, M	are the normal load and bending moment;
ω	is the angular velocity in the plastic hinge;
b, h	are the width and height of rectangular cross sections;
ξ	is the coordinate of the neutral axis;
$R, \Delta R$	are the radius of the axis of rigidity of the ring and its increment;
k	is the geometric hardening coefficient.

LITERATURE CITED

1. L. M. Kachanov, Fundamentals of the Theory of Plasticity [in Russian], Gostekhizdat, Moscow (1956).
2. W. Prager, Problems in the Theory of Plasticity [Russian translation], Fizmatgiz, Moscow (1958).
3. V. N. Zalesov, Proc. 6th All-Union Conf. Theory of Shells and Plates [in Russian], Nauka, Moscow (1967), pp. 404-409.